

**AFCRC-TN-58-154**

**ASTIA Document No. AD 152385**

INSTITUTE OF MATHEMATICAL SCIENCES

LE...

25 Waverly Place, New York 3, N.Y.



**NEW YORK UNIVERSITY**

Institute of Mathematical Sciences

Division of Electromagnetic Research

**RESEARCH REPORT No. EM-112**

# **An Integral Representation of the Electromagnetic Field in the Image Space of an Optical System**

**EMIL WOLF**

**CONTRACT No. AF 19(604)1717**

**MARCH, 1958**



NEW YORK UNIVERSITY

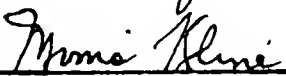
Institute of Mathematical Sciences  
Division of Electromagnetic Research

Research Report No. EM-112

AN INTEGRAL REPRESENTATION OF THE ELECTROMAGNETIC  
FIELD IN THE IMAGE SPACE OF AN OPTICAL SYSTEM

E. Wolf

  
E. Wolf

  
Morris Kline  
Project Director

The research reported in this document has been sponsored by the Air Force Cambridge Research Center, Air Research and Development Command, under Contract No. AF 19(604)1717.



Abstract

Preliminary to the study of the structure of an optical image from the standpoint of electromagnetic theory, an integral representation is obtained, for the electromagnetic field in the image space of an optical system. This representation, which is not restricted to systems of low angular aperture, is in the form of an angular spectrum of plane waves, and is closely related to that introduced by Luneberg (1944) as a vector generalization of well-known formulas of Debye (1909) and Picht (1925). It is shown that the representation has a simple physical interpretation in terms of a modified Huygens-Fresnel principle which operates with secondary plane waves rather than with secondary spherical waves.

Table of Contents

	<u>Page</u>
1. Introduction	1
2. An integral representation of the image field	2
3. A physical interpretation of the diffraction integrals	10
Appendix	15
References	18
Distribution List	



## 1. Introduction

Practically all investigations which relate to the structure of the optical image have, up to now, been almost exclusively based on geometrical optics or on the scalar diffraction theory. Valuable as these methods are in connection with the design of optical systems and the evaluation of their performance\*, they suffer from two limitations, namely: (1) they do not give indications of the vectorial features of the image, i.e., the state of polarization of the light in the image region and the direction of the energy flow (Poynting vector); and (2) they cannot be easily applied to the problem of determining the structure of the image in systems of wide angular aperture, such as certain photographic lenses and some modern astronomical cameras. It would appear that more refined methods are needed for the treatment of such problems. Development of more powerful methods also seems desirable because of problems which arise in the related field of optics of microwaves. Microwave systems frequently employ numerical apertures which are considerably larger than those encountered in systems transmitting light, and it is not always clear to what extent the usual methods can be used with confidence [cf. Bachynski and Bekefi, 1956]. Moreover, in microwave optics the knowledge of the state of polarization of the image field is of direct interest.

In the present investigation an integral representation is obtained for the electromagnetic field in the image space of an optical system, which makes it possible to determine the complete structure of the image in systems of low as well as high angular aperture. Since the specification of the exact boundary conditions which the image field satisfies present formidable difficulties, the representation is an approximate one, but it may be expected to predict the field with a high degree of accuracy at all points which are not too close to the exit pupil. (The distance from the aperture must be large compared to the wavelength.) This is the case of the main interest in connection with practical applications. Our representation is intimately connected with that of Luneberg (1944) and may be regarded as a generalization of the well-known representation introduced for the scalar case by Debye (1909) and extended by Picht (1925). The present derivation appears to be

---

\*For review of the literature see Wolf (1951). A survey of recent research on the foundations of some of the methods is given in Wolf (1955).

simpler than Luneberg's and shows clearly the nature of the approximations that are involved when the solution is applied to problems of practical interest. It is also shown that the representation has a simple physical interpretation in terms of a modified Huygens-Fresnel principle which operates with (vectorial) secondary plane waves, rather than with secondary spherical waves.

In a subsequent paper the results will be used to determine the structure of electromagnetic images in the focal plane of aplanatic systems of low as well as high angular aperture.

## 2. An integral representation of the image field

Let  $S_0$  be a monochromatic point source of angular frequency  $\omega$ , situated in the object space of a rotationally symmetrical system and let

$$(2.1) \quad \vec{E}(P,t) = \Re \vec{e}(P)e^{-i\omega t}, \quad \vec{H}(P,t) = \Re \vec{h}(P)e^{-i\omega t}$$

denote the electric and magnetic vectors, at a point  $P$  in the image space, at time  $t$ . The vectors  $\vec{e}$  and  $\vec{h}$  which are to be determined are in general complex, and the symbol  $\Re$  denotes the real part.

Let  $\vec{s}$  denote a unit vector along a typical geometrical ray in the image field, i.e., a ray which reaches the image space through the exit pupil of the system. The positive direction of  $\vec{s}$  is taken in the direction of the propagation of the light (Figure 1). We shall show that within an accuracy adequate for practical purposes,  $\vec{e}(P)$  and  $\vec{h}(P)$  for any point  $P$  in the image space which is not too close to the plane of the exit pupil may be expressed in the form

$$(2.2) \quad \vec{e}(P) = \iint_{\Omega} \vec{A}(\vec{s}) e^{ik\vec{s} \cdot \vec{r}} d\Omega, \quad \vec{h}(P) = \iint_{\Omega} \vec{B}(\vec{s}) e^{ik\vec{s} \cdot \vec{r}} d\Omega.$$

Here  $\vec{r}$  is the position vector of  $P$ ,  $\vec{A}(\vec{s})$  and  $\vec{B}(\vec{s})$  are certain (generally complex) vector functions of  $\vec{s}$  which are specified by the geometric optics field, and the integration is taken over the solid angle  $\Omega$  formed by all the geometrical rays which reach the image space. The symbol  $k$  is the wave number in the image, i.e.,

$$(2.3) \quad k = n \frac{\omega}{c} = \frac{2\pi}{\lambda} = n \frac{2\pi}{\lambda_0} = n k_0,$$

where  $n$  is the refractive index of the image space,  $c$  is the vacuum wavelength,  $\lambda$  and  $\lambda_0$  are the wavelengths in the image space and in the vacuum respectively, and  $k_0$  is the vacuum wave number.



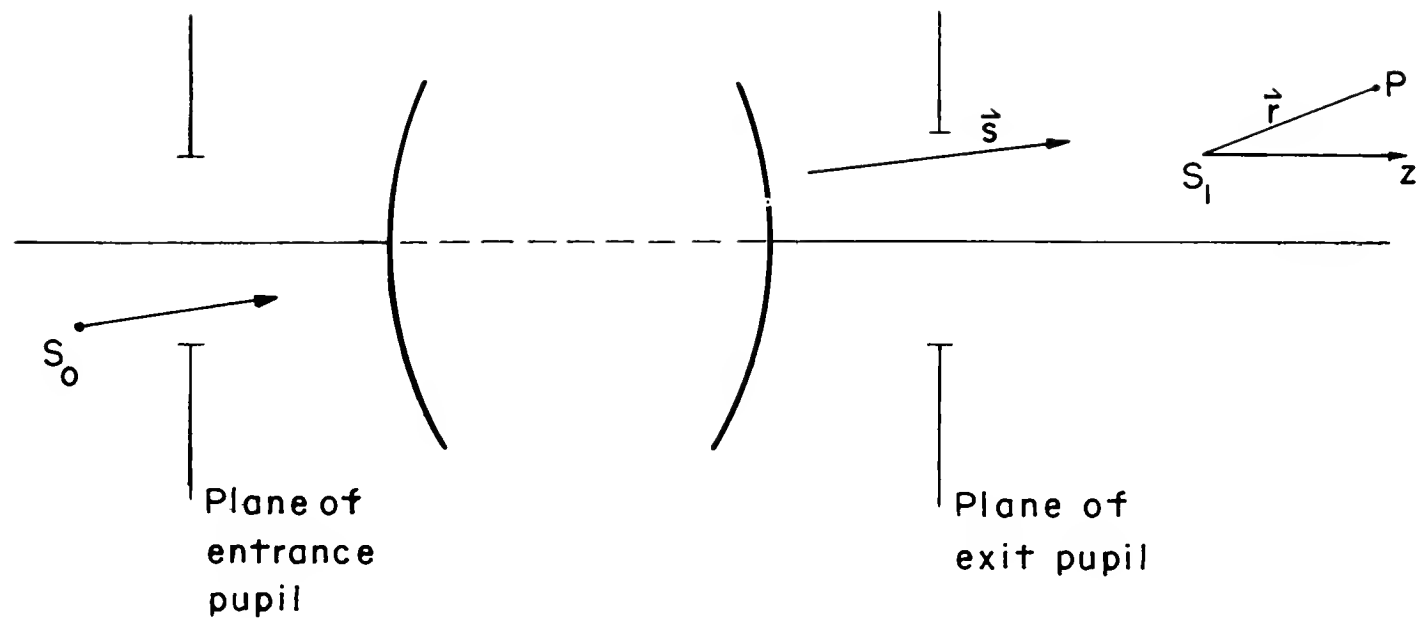


Figure 1. Illustrating the notation

The integrals (2.2) represent the image field in the form of an angular spectrum of plane waves. A representation of this type is always possible provided that the unit vector  $\vec{s}$  spans all directions including complex ones [cf. Bouwkamp (1954), 41-42]. These latter directions represent surface waves or evanescent waves. For our purposes it will be sufficient to include only the real directions which are associated with rays in the image field of the instrument ( $\vec{s}$  belonging to  $\Omega$ ).

Let  $x, y, z$  be the components of the position vector  $\vec{r}$  (coordinates of  $P$ ) with respect to Cartesian rectangular axes with origin at the Gaussian image point  $S_1$  of  $S_0$  and with the positive  $z$ -direction at right angles to the plane of the exit pupil and pointing away from it (Figure 1). Further, let  $s_x, s_y, s_z$  be the components of  $\vec{s}$ . Since  $\vec{s}$  is a unit vector, and the light is propagated into the image space ( $s_z > 0$ ),

$$(2.4) \quad s_z = \left| \sqrt{1 - s_x^2 - s_y^2} \right|.$$

The element  $d\Omega$  of the solid angle is given by\*

$$(2.5) \quad d\Omega = \frac{ds_x ds_y}{s_z},$$

so that the first integral in (2.2) may be written as

$$(2.6) \quad \vec{e}(P) = \iint_{\Omega} \vec{A}(s_x, s_y) \exp[ik(s_x x + s_y y + s_z z)] \frac{ds_x ds_y}{s_z};$$

there is, of course, a strictly similar expression for  $\vec{h}(P)$ .

\*This follows, on transforming the usual expression

$$(I) \quad d\Omega = \sin \theta \cos \theta d\theta d\phi$$

where  $(\theta, \phi)$  are the spherical polar angles. We have  $s_x = \sin \theta \cos \phi$ ,  $s_y = \sin \theta \sin \phi$ ,  $s_z = \cos \theta$ . Hence

$$(II) \quad ds_x ds_y = \begin{vmatrix} \frac{\partial s_x}{\partial \theta} & \frac{\partial s_x}{\partial \phi} \\ \frac{\partial s_y}{\partial \theta} & \frac{\partial s_y}{\partial \phi} \end{vmatrix} d\theta d\phi$$

$$= \sin \theta \cos \theta d\theta d\phi = s_z \sin \theta d\theta d\phi,$$

and (2.5) follows from (I) and (II).

To determine the vectors  $\vec{A}$  and  $\vec{B}$ , consider first the behavior of the integral (2.6) when the distance

$$(2.7) \quad r = |\vec{r}| = \left| \sqrt{x^2 + y^2 + z^2} \right|$$

of the point of observation P from the origin of coordinates  $S_1$  is very many wavelengths. Let  $\vec{u}(u_x, u_y, u_z)$  be the unit vectors in the direction  $S_1P$ , i.e.,

$$(2.8) \quad u_x = \frac{x}{r}, \quad u_y = \frac{y}{r}, \quad u_z = \frac{z}{r} = \sqrt{1 - u_x^2 - u_y^2}.$$

Using (2.7) and (2.8), the integral (2.6) may be written in the form

$$(2.9) \quad \vec{e}(P) = \iint_{\Omega} \frac{\vec{A}(s_x, s_y)}{s_z} \exp[ikr \phi(s_x, s_y; u_x, u_y)] ds_x ds_y,$$

where

$$(2.10) \quad \begin{aligned} \phi(s_x, s_y; u_x, u_y) &= s_x u_x + s_y u_y + s_z u_z \\ &= s_x u_x + s_y u_y + \left| \sqrt{1 - s_x^2 - s_y^2} \right| \sqrt{1 - u_x^2 - u_y^2}. \end{aligned}$$

Now when  $kr = 2\pi r/\lambda \gg 1$ , the exponential term in (2.9) varies rapidly as  $(s_x, s_y)$  explores the domain of integration, and the real and imaginary parts will change sign many times. In consequence the contribution to the integral from the different elements of the domain of integration virtually cancel out; the only appreciable contribution comes from the elements (if any) centered on  $(s_x, s_y)$  in the domain of integration for which  $\phi$  is stationary. This is the essence of the principle of stationary phase which may be used to find the asymptotic approximation to (2.9) for large values of  $kr$ . This approximation is derived in the Appendix at the end of this paper. It is shown there, that when  $z < 0$ , i.e., when P lies on the same side of the focal plane as the exit pupil, then

$$(2.11) \quad \begin{aligned} \vec{e}(P) &= -\frac{2\pi}{ik} \vec{A}(-u_x, -u_y) \frac{e^{-ikr}}{r} + \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} \quad \text{when } -\vec{u} \in \Omega, \\ &= \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} \quad \text{when } -\vec{u} \notin \Omega. \end{aligned}$$

The symbols  $\in$  and  $\notin$  stand for 'belongs to' and 'does not belong to' respectively, i.e., the first line in (2.11) applies when the direction  $PS_1$  is parallel to one

of the geometrical rays which reaches the image space, and the second line applies when there is no such ray. When  $z > 0$ , i.e., when  $P$  lies on the side of the focal plane opposite to that of the exit pupil, one has, in place of (2.11),

$$(2.12) \quad \vec{e}(P) = \frac{2\pi}{ik} \vec{A}(u_x, u_y) \frac{e^{ikr}}{r} + \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} \quad \text{when } \vec{u} \in \Omega ,$$

$$= \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} \quad \text{when } \vec{u} \notin \Omega .$$

Now the exact boundary conditions which the field vectors satisfy cannot be determined in practice. However, if the linear dimensions of the aperture are large compared to the wavelength, and if  $P$  is not too close to the aperture, a good approximation to  $\vec{e}$  and  $\vec{h}$  is obtained by imposing the approximate Kirchhoff's boundary conditions, i.e., the conditions

$$(2.13) \quad \vec{e}(P') = \vec{e}^{(i)}(P'), \quad \vec{h}(P') = \vec{h}^{(i)}(P') \quad \text{when } P' \in \mathcal{A} ,$$

$$\vec{e}(P') = 0 , \quad \vec{h}(P') = 0 \quad \text{when } P' \notin \mathcal{A} ,$$

where  $\vec{e}^{(i)}(P')$ ,  $\vec{h}^{(i)}(P')$  are the values of the unperturbed incident field at a point  $P'$  in the plane of the exit pupil  $\mathcal{A}$ . If we neglect terms of order  $(kr)^{-3/2}$  we have from (2.11) and (2.13)

$$(2.14) \quad \vec{A}(s_x, s_y) = -\frac{ik}{2\pi} \vec{e}^{(i)}(P') r' e^{ikr'} ,$$

where  $P'$  is the point in the direction  $\vec{u} = -\vec{s}$  from  $S_1$ , and  $r' \gg \lambda$  is the distance  $S_1 P'$ . Within the accuracy here in question  $P'$  may evidently be also identified with the point in the aperture through which the ray with direction of propagation  $\vec{s} = -\vec{u}$  passes.

Except for points  $P'$  in the immediate neighborhood of the edge of the aperture,  $\vec{e}^{(i)}(P')$  is, to a good approximation, given by [cf. Born and Wolf, 1958, Chapter III]

$$(2.15) \quad \vec{e}^{(i)}(P') = \frac{\vec{a}}{\sqrt{R_1} \sqrt{R_2}} \exp\left[ik_0 \mathcal{S}(P')\right] ,$$

---

\*In a typical case in light optics  $r' \gtrsim 10$  cm,  $\lambda \sim 5 \cdot 10^{-5}$  cm so that  $kr' = 2\pi r' / \lambda \gtrsim 10^6$ .

where  $R_1$  and  $R_2$  are the principal radii of curvature of the geometrical wave front through  $P'$ ,  $\mathcal{S}(P')$  is the eikonal function and  $\vec{a}$  is a (generally complex) vector which is constant along the ray and perpendicular to it. We shall call  $\vec{a}$  the strength factor of the ray ( $s_x, s_y$ ). With a suitable choice of the phase constant of  $\vec{a}$ ,  $\mathcal{S}(P')$  represents the optical path from the source point  $S_0$  to  $P'$ . From (2.14) and (2.15)

$$(2.16) \quad \vec{A}(s_x, s_y) = -\frac{ik}{2\pi} \frac{r'}{\sqrt{R_1} \sqrt{R_2}} \exp \left[ ik_0 \left\{ \mathcal{S}(P') + nr' \right\} \right] .$$

We may express (2.14) in a somewhat different form which involves the wave aberration function of the system. Let  $\sigma$  be the Gaussian reference sphere, i.e., a sphere which has its center at the Gaussian image point  $S_1$  of  $S_0$  and which passes through the center  $E$  of the exit pupil. Further, let  $W$  be the geometrical wave front through  $E$ . If  $F$  and  $G$  are the points in which  $P'S_1$  intersects  $\sigma$  and  $W$  respectively, we have (see Figure 2),

$$(2.17) \quad \mathcal{S}(P') + nr' = [S_0G] + [GF] + [FS_1] ,$$

where the brackets [...] denote optical lengths. Now since  $G$  and  $E$  lie on the same wave front,  $[S_0G] = [S_0E]$ . Also  $[FS_1] = nR$  where  $R$  is the radius of the reference sphere. Hence

$$(2.18) \quad \mathcal{S}(P') + nr' = n(C + \Phi) ,$$

where

$$(2.19) \quad C = \frac{1}{n} [S_0E] + R = \text{constant}$$

and

$$(2.20) \quad \Phi = \Phi(s_x, s_y) = \frac{1}{n} [GF]$$

is the aberration function\* of the system. Also, since the principal centers of curvature of the geometrical wave front lie in the region of the optical image,  $R_1$ ,  $R_2$  and  $r'$  can only differ by small relative amounts, so that the

---

\*The aberration is usually measured along the ray through  $F$  rather than along the radial direction  $SP_1$ . For practical purposes the two definitions may be taken as equivalent [cf. Wolf, 1952].

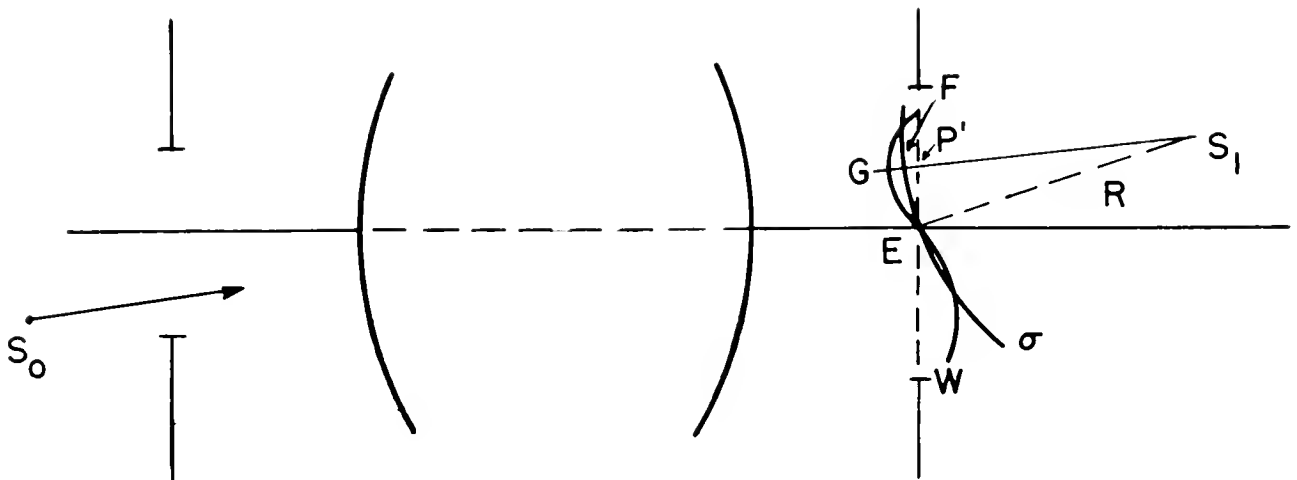


Figure 2. Definition of the aberration function.

term  $r'/\sqrt{R_1}\sqrt{R_2}$  in (2.16) may be replaced by unity. If we also substitute from (2.18), (2.16) becomes

$$(2.21) \quad \vec{A}(s_x, s_y) = -\frac{ik}{2\pi} \vec{a}(s_x, s_y) \exp\left[ik\left\{C + \Phi(s_x, s_y)\right\}\right],$$

and the diffraction integral (2.6) finally takes the form\*

$$(2.22) \quad \vec{e}(x, y, z) = -\frac{ik}{2\pi} e^{ikC} \iint_{\Omega} \frac{\vec{a}(s_x, s_y)}{s_z} \exp\left[ik\left\{\Phi(s_x, s_y) + s_x x + s_y y + s_z z\right\}\right] ds_x ds_y.$$

In a strictly similar manner the corresponding expression for  $\vec{h}$  is obtained:

$$(2.23) \quad \vec{h}(x, y, z) = -\frac{ik}{2\pi} e^{ikC} \iint_{\Omega} \frac{\vec{b}(s_x, s_y)}{s_z} \exp\left[ik\left\{\Phi(s_x, s_y) + s_x x + s_y y + s_z z\right\}\right] ds_x ds_y.$$

Here  $\vec{b}$  is the magnetic strength factor, i.e., the vector which bears the same relation to the incident magnetic field  $\vec{h}^{(i)}$  as  $\vec{a}$  bears to  $\vec{e}^{(i)}$  [cf. (2.15)]. It follows from Maxwell's equations that  $\vec{a}$  and  $\vec{b}$  obey the relation [cf. Born and Wolf (1958), § 3.1 eq. (12a) and (15a)]

$$(2.24) \quad \vec{b} = \sqrt{\epsilon/\mu} \vec{s} \wedge \vec{a},$$

where  $\epsilon$  and  $\mu$  denote the dielectric constant and the magnetic permeability of the image space, and the relation  $n = \sqrt{\epsilon\mu}$  is assumed. In our system of units (Gaussian)  $\mu$  differs inappreciably from unity, but is retained here for the sake of symmetry.

The formulas (2.22) and (2.23) represent the solution of our problem. They express the image field as a superposition of plane waves of different directions of propagation. The 'complex amplitudes'  $\vec{a}$ ,  $\vec{b}$  of each constituent plane wave can be derived from a ray trace, taking into account the state of polarization of the field along each ray. The phase function  $\Phi$ , which is a measure of the aberration of the system, may likewise be determined from the ray trace. It follows, in view of (2.24), that our solution satisfies the

---

\*Since the origin of the phase is arbitrary, we may, of course, set  $C = 0$ .

homogeneous Maxwell's equations

$$(2.25) \quad \begin{aligned} \text{curl } \vec{h} + ik_0 \epsilon \vec{e} &= 0, \\ \text{curl } \vec{e} - ik_0 \mu \vec{h} &= 0, \end{aligned}$$

throughout the whole image space, and it is seen from (2.12) that it does obey the vectorial form of the Sommerfeld radiation condition [cf. C. Müller (1957), p. 13-14] in the half-space  $z > 0$ . Since the approximate Kirchhoff boundary conditions were used, our solution cannot be expected to represent the field adequately at points in the immediate neighborhood of the exit pupil. However, when the aperture is large compared to the wavelength, and when the point of observation  $P(x,y,z)$  as well as the Gaussian focus  $S_1$  are both situated at distances from the aperture that are also large compared to the wavelength, (2.22) and (2.23) may be expected to give a good approximation to the field.

### 3. A physical interpretation of the diffraction integrals

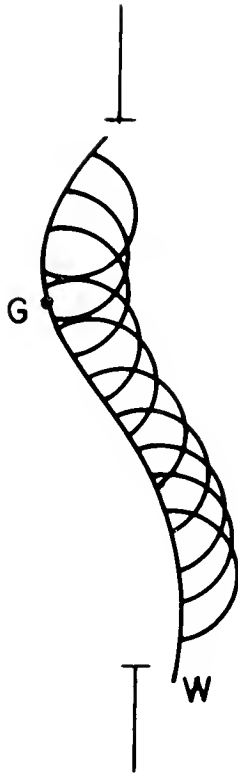
The integrals (2.22) and (2.23) have a simple physical interpretation which is clearly brought out when comparing them with the corresponding integrals based on the application of the Huygens-Fresnel principle. According to the Huygens-Fresnel principle, each point  $G$  of the wave front  $W$  may be regarded as a center of a secondary disturbance which is propagated as a spherical wave of complex amplitude proportional to  $\vec{e}^{(i)}(G)$ ,  $\vec{h}^{(i)}(G)$ . The total field at  $P$  is obtained by adding the effects of all the spherical waves which reach the image space through the exit pupil. Thus the Huygens-Fresnel approximation (suffix HF) for the field is

$$(3.1) \quad \begin{aligned} \vec{e}_{\text{HF}}(P) &= \iint_W \vec{e}^{(i)}(G) \frac{e^{ikd}}{d} K(G) dW, \\ \vec{h}_{\text{HF}}(P) &= \iint_W \vec{h}^{(i)}(G) \frac{e^{ikd}}{d} K(G) dW, \end{aligned}$$

where  $d$  is the distance  $GP$  and  $K$  is the usual inclination factor, which, for small obliquities, has the constant value  $-ik/2\pi$ . The integration extends over the part of the wave front which approximately fills the exit pupil (Figure 3(a)).

In the present formulation, the field is given by

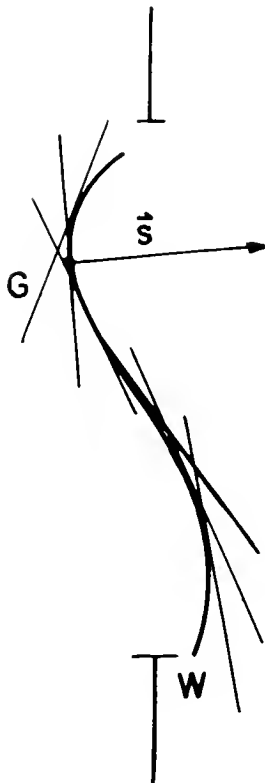




(a) Huygens' construction relating to the integrals (3.1).

Spherical Secondary Waves

$$\vec{e}^{(i)}(G) \frac{e^{ikd}}{d}, \quad \vec{h}^{(i)}(G) \frac{e^{ikd}}{d}.$$



(b) A construction relating to the integrals (3.2).

Plane Secondary Waves

$$\vec{A}(\vec{s}) e^{ik\vec{s} \cdot \vec{r}}, \quad \vec{B}(\vec{s}) e^{ik\vec{s} \cdot \vec{r}}.$$

Figure 3. Physical interpretation of the diffraction integrals.

$$(3.2) \quad \begin{aligned} \vec{e}(P) &= \iint_{\Omega} \vec{A}(\vec{s}) e^{ik\vec{s} \cdot \vec{r}} d\Omega, \\ \vec{h}(P) &= \iint_{\Omega} \vec{B}(\vec{s}) e^{ik\vec{s} \cdot \vec{r}} d\Omega, \end{aligned}$$

i.e., as superposition of plane waves of complex amplitudes  $\vec{A}(\vec{s})$ ,  $\vec{B}(\vec{s})$ , propagated in the  $\vec{s}$ -directions coinciding with the directions of all the geometrical rays that reach the image space. To each plane wave there belongs a plane wave front which is tangential to the wave front  $W$  at the appropriate point  $G$  (see Figure 3(b)). Thus each point of that portion of the wave front  $W$  which approximately fills the exit pupil may be regarded as giving rise to a secondary plane wave, and the field at  $P$  is obtained by adding the effects of all these secondary waves. If terms of order  $(1/kr)^{3/2}$  are neglected, the vector amplitudes in the two constructions are, according to (2.11), related by

$$(3.3) \quad \begin{aligned} \vec{e}^{(i)}(G) &= -\frac{2\pi}{ik} \vec{A}(\vec{s}) \frac{e^{-ikr}}{r}, \\ \vec{h}^{(i)}(G) &= -\frac{2\pi}{ik} \vec{B}(\vec{s}) \frac{e^{-ikr}}{r}, \end{aligned}$$

where  $r$  is the distance  $S_1 G$ .

Our diffraction integrals (2.22) and (2.23) are closely related to the diffraction integrals of Luneberg (1944) and may be regarded as generalizations of the integral introduced by Debye (1909) to represent a diffracted spherical scalar wave. A generalization of Debye's representation to non-spherical (scalar) waves was carried out by Picht (1925); this case was also treated recently by Focke (1956), who employed essentially the same method as used in the present paper.

Luneberg's integrals may be derived from our solution as follows: We have from (2.19) and (2.20),

$$(3.4) \quad k \{C + \Phi\} = k_0 \left\{ [S_0 E] + [GF] + [FS_1] \right\}.$$

Let  $N$  be the foot of the perpendicular from  $S_1$  onto the ray through  $G$  (Figure 4). In the asymptotic approximation,  $[GF] + [FS_1]$  may evidently be replaced by the optical length  $[GN]$  measured along the ray. Further, since  $G$  and  $E$  lie on the same wave front,  $[S_0 E] = [S_0 G]$ , and we may write in place of (3.4)

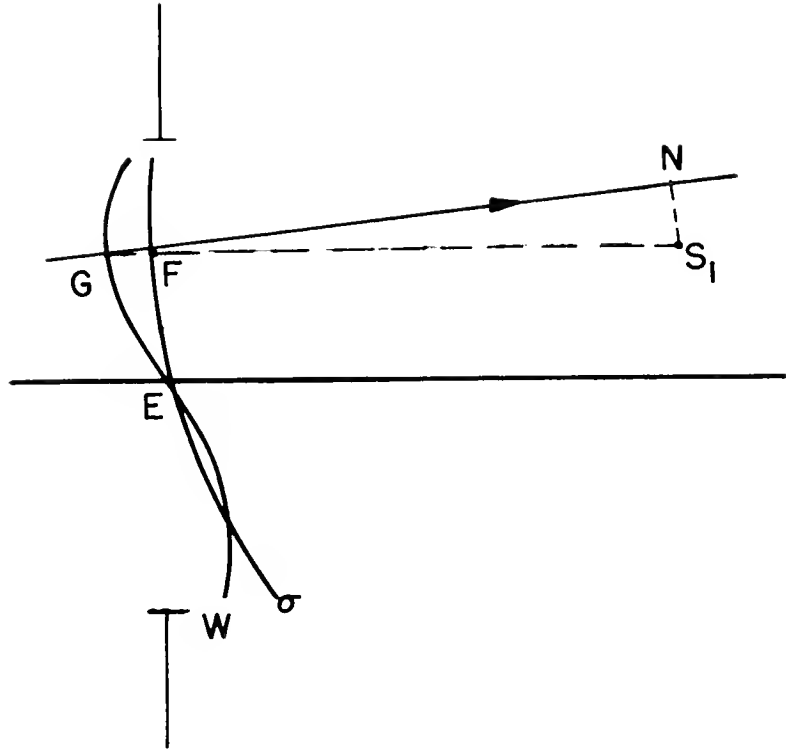


Figure 4: Transition to Luneberg's diffraction integrals.

$$(3.5) \quad k \{C + \Phi\} = k_o \{[S_o G] + [GN]\} = k_o [S_o N] .$$

Let us now introduce in place of the unit vector  $\vec{s}$  the 'ray vector'

$$(3.6) \quad \vec{g} = n \vec{s} ,$$

and let  $p$ ,  $q$  and  $m$  be its components, i.e.,

$$(3.7) \quad p = ns_x, \quad q = ns_y, \quad m = ns_z = \left| \sqrt{n^2 - p^2 - q^2} \right| .$$

Then  $[S_o N]$  becomes a function of  $p$  and  $q$ , and of the position of the source point  $S_o$ ; it is the mixed characteristic function  $W(S_o; p, q)$  of Hamilton. The strength factors  $\vec{a}$  and  $\vec{b}$ , regarded as functions of  $p$  and  $q$  rather than of  $s_x$

and  $s_y$ , will be denoted by  $\tilde{a}$  and  $\tilde{b}$  respectively, and the corresponding domain of integration, formed by the  $(p,q)$ -components of all the geometrical rays which reach the image space, will be denoted by  $\tilde{\Omega}$ . The integrals (2.22) and (2.23) become

$$(3.8) \quad \vec{e}(x,y,z) = -\frac{ik_0}{2\pi} \iint_{\tilde{\Omega}} \frac{\tilde{a}(p,q)}{m} \exp\left[ik_0\{W(S_0;p,q) + px+qy+mz\}\right] dpdq ,$$

$$(3.9) \quad \vec{h}(x,y,z) = -\frac{ik_0}{2\pi} \iint_{\tilde{\Omega}} \frac{\tilde{b}(p,q)}{m} \exp\left[ik_0\{W(S_0;p,q) + px+qy+mz\}\right] dpdq .$$

In place of (2.24) we now have

$$(3.10) \quad \tilde{b} = \frac{1}{\mu} \vec{g} \wedge \tilde{a} .$$

The formulas (3.8) and (3.9) are formally identical with Luneberg's diffraction integrals. They were obtained by Luneberg not as an approximate solution of the true physical problem, but rather as an exact solution to an idealized problem, namely the following: to find a solution of the homogeneous Maxwell's equations, valid throughout the whole space, which has a prescribed behavior at infinity. In spite of the mathematical elegance of Luneberg's formulation, we prefer to regard our integrals as an approximate solution of the true physical problem, valid at distances from the aperture that are large compared to the wavelength. Our analysis shows that the integrals (2.22) and (2.23) [or (3.8) and (3.9)] are then effectively equivalent to the solution obtained by a direct application of the (vectorial) Huygens-Fresnel-Kirchhoff diffraction theory. However, since our formulas do not involve an inclination factor, they are more suitable for the treatments of problems involving large obliquities, e.g., the problem of determining the structure of an optical image in a system with a high angular aperture. This problem will be discussed in another paper.

# Appendix

## Asymptotic evaluation of the integral (2.9)

We derive here the asymptotic approximation for large  $kr$  to the integral (2.9), viz.,

$$(A1) \quad \vec{e} = \iint_{\Omega} \frac{\vec{A}(s_x, s_y)}{s_z} \exp[ikr\phi(s_x, s_y; u_x, u_y)] ds_x ds_y ,$$

where

$$(A2) \quad \phi(s_x, s_y; u_x, u_y) = s_x u_x + s_y u_y + s_z u_z ,$$

$$(A3) \quad s_z = \left| \sqrt{1 - s_x^2 - s_y^2} \right| ,$$

$$(A4) \quad u_z = \sqrt{1 - u_x^2 - u_y^2} .$$

As explained in Section 2, the asymptotic approximation may be obtained by the application of the principle of stationary phase. The principle is well known in connection with single integrals; its generalization to double integrals of a wide class of practical interest was carried out by Focke (1954), who determined the whole asymptotic expansion of these integrals. Other ways of deriving the asymptotic expansions were discussed by Jones and Kline (1956), and Chako (1958). It follows from these investigations, that for large  $kr$ , (A1) reduces to\*

$$(A5) \quad \vec{e} = \frac{2\pi i}{kr} \sum \frac{1}{\sqrt{|\Delta|}} \frac{\vec{A}(s'_x, s'_y)}{s'_z} \exp[ikr\phi(s'_x, s'_y; u_x, u_y)] + \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} .$$

Here  $s'_x, s'_y$  are the values of  $s_x$  and  $s_y$  for which  $\phi$  is stationary within the domain of integration, i.e., they are the roots of the equation

$$(A6) \quad \frac{\partial \phi}{\partial s_x} = \frac{\partial \phi}{\partial s_y} = 0 ;$$

---

\*Strictly speaking, the 'amplitude function' in the integrals considered by these authors is a real scalar, not a complex vector as in (A1), but the generalization to the present case is quite trivial.

further,

$$(A7) \quad \Delta = \left[ \frac{\partial^2 \phi}{\partial s_x^2} \frac{\partial^2 \phi}{\partial s_y^2} - \left( \frac{\partial^2 \phi}{\partial s_x \partial s_y} \right)^2 \right]',$$

$$(A8) \quad j = \begin{cases} 1 & \text{when } \Delta > 0, \quad \left( \frac{\partial^2 \phi}{\partial s_x^2} \right)' > 0, \\ -1 & \text{when } \Delta > 0, \quad \left( \frac{\partial^2 \phi}{\partial s_x^2} \right)' < 0, \\ i & \text{when } \Delta < 0, \end{cases}$$

and the prime in (A7) and (A8) denotes values at the stationary point. The summation in (A5) is taken over all the stationary points.

If there is no stationary point within the domain  $\Omega$  of integration, then

$$(A9) \quad \vec{e} = \mathcal{O}\left(\frac{1}{kr}\right)^{3/2}.$$

Now we have from (A2) and (A3),

$$(A10) \quad \frac{\partial \phi}{\partial s_x} = u_x + u_z \frac{\partial s_z}{\partial s_x} = u_x - u_z \frac{s_x}{s_z},$$

with a similar expression for  $\partial \phi / \partial s_y$ . It follows, if (A1) is also used, that  $\phi$  is stationary when

$$(A11) \quad s_x' = u_x, \quad s_y' = u_y, \quad s_z' = u_z \quad \text{if } u_z > 0, \quad (a)$$

$$s_x' = u_x, \quad s_y' = u_y, \quad s_z' = u_z \quad \text{if } u_z < 0. \quad (b)$$

Thus, if  $u_z > 0$ , there is one stationary point which contributes to the asymptotic approximation of  $\vec{e}$ , provided that  $\vec{u}$  lies within  $\Omega$ ; if  $u_z < 0$ , the same is true, provided that  $-\vec{u}$  lies in  $\Omega$ . If  $u_z$  (or  $-u_z$ , as the case may be) does not lie in  $\Omega$ ,  $\phi$  is not stationary within the domain of integration, and, by (A9)  $\vec{e}$  is then of the order of  $(kr)^{-3/2}$ .

From (A10) and the corresponding relation for  $\partial\phi/\partial s_y$ , we have, on using (A11),

$$(A12) \quad \left(\frac{\partial^2 \phi}{\partial s_x^2}\right)' = \mp \left(1 + \frac{u_x^2}{u_z^2}\right), \quad \left(\frac{\partial^2 \phi}{\partial s_y^2}\right)' = \mp \left(1 + \frac{u_y^2}{u_z^2}\right), \quad \left(\frac{\partial^2 \phi}{\partial s_x \partial s_y}\right)' = \mp \frac{u_x u_y}{u_z^2},$$

where the upper sign is taken when  $u_z > 0$  and the lower sign, when  $u_z < 0$ . The expressions (A7) and (A8) become

$$(A13) \quad \Delta = \frac{1}{u_z^2},$$

$$(A14) \quad j = \mp 1 \quad \text{according as } u_z \gtrless 0.$$

We also have

$$(A15) \quad \vec{A}(s'_x, s'_y) = \vec{A}(\pm u_x, \pm u_y),$$

$$(A16) \quad \phi(s'_x, s'_y; u_x, u_y) = \pm 1,$$

the upper or lower sign again being taken according as  $u_z \gtrless 0$ .

From (A5) and (A9) we thus finally obtain the following asymptotic approximations to  $\vec{e}$  for large  $kr$ :

$$(A17) \quad \left. \begin{aligned} \vec{e} &= \frac{2\pi}{ik} \vec{A}(u_x, u_y) \frac{e^{ikr}}{r} + \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} && \text{when } u_z > 0 \text{ and } \vec{u} \in \Omega, \\ &= \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} && \text{when } u_z > 0 \text{ and } \vec{u} \notin \Omega, \end{aligned} \right\} (a)$$

$$\left. \begin{aligned} \vec{e} &= -\frac{2\pi}{ik} \vec{A}(-u_x, -u_y) \frac{e^{-ikr}}{r} + \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} && \text{when } u_z < 0 \text{ and } -\vec{u} \in \Omega, \\ &= \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} && \text{when } u_z < 0 \text{ and } -\vec{u} \notin \Omega. \end{aligned} \right\} (b)$$

References

- Bachynski, M.P. and G. Bekefi (1956) - Aberrations in circularly symmetric microwave lenses; Trans. IRE; AP-4, 412.
- Born, M. and E. Wolf (1958) - Principles of Optics; London, Pergamon Press. (In press.)
- Bouwkamp, C.J. (1954) - Diffraction theory; (London: The Physical Society); Rep. Progr. Phys., 17, 35.
- Chako, N. (1958) - On the evaluation of double integrals containing a large parameter. (In press.)
- Debye, P. (1909) - Das Verhalten von Lichtwellen in der Nähe eines Brennpunktes oder einer Brennnlinie; Ann. d. Phys. (Leipzig); 30, 755.
- Focke, J. (1954) - Asymptotische Entwicklungen der Methode der stationären Phase; Ber. Sächs. Akad. Wiss., Leipzig; 101, heft 3.
- Focke, J. (1956) - Wellen-optische Untersuchungen zum Öffnungsfehler; Optica Acta; 3, 110.
- Jones, D.S. and M. Kline (1956) - Asymptotic expansion of multiple integrals and the method of stationary phase; N.Y.U., Inst. Math. Sci., Div. EM Res., Research Report No. EM-100, October, 1956. (In press: J. Math. Phys.)
- Luneberg, R.K. (1944) - Mathematical theory of optics; lecture notes, Brown University.
- Müller, C. (1957) - Grundprobleme der Mathematischen Theorie Electromagnetischer Schwingungen; Berlin-Göttingen-Heidelberg; Springer-Verlag.
- Picht, J. (1925) - Über den Schwingungsvorgang der einen beliebigen (astigmatischen) Strahlenbündel entspricht; Ann. d. Phys. (Leipzig); 77, 685.



- Wolf, E. (1951) - The diffraction theory of aberrations; (London: The Physical Society); Rep. Progr. Phys.; 14, 95.
- Wolf, E. (1952) - On a new aberration function of optical instruments; J. Opt. Soc. Amer.; 42, 547.
- Wolf, E. (1955) - Recent researches on the foundations of geometrical optics and related investigations in electromagnetic theory; Trans. IRE; AP-3, 228.



DISTRIBUTION LIST FOR RESEARCH REPORTS

Contract No. AF 19(604)1717

(ONE copy unless otherwise noted)

Commander Air Force Missile Test Center Patrick Air Force Base, Florida ATTN: Technical Library	Director Evans Signal Laboratory Belmar, New Jersey ATTN: Technical Document Center	Chief, Bureau of Aeronautics Department of the Navy Washington 25, D.C. ATTN: Code Aer-TD-414
Commander Air Research and Development Command Washington 25, D.C. ATTN: RDDI	Massachusetts Institute of Technology Signal Corps Liaison Officer Cambridge 39, Massachusetts ATTN: A.D. Bedrosian, Room 20C-116	Chief, Bureau of Ships Navy Department Washington 25, D.C. ATTN: Mr. E. Johnston, Code 938
Commander Air Research and Development Command Washington 25, D.C. ATTN: Major E. Wright, RDTDR-1	Commanding General SC Engineering Laboratories Fort Monmouth, New Jersey ATTN: Sceltechnical Documents Center Bay 2C122	Commander U.S. Naval Air Missile Test Center Point Mugu, California ATTN: Code 366, MT-6
Director Air University Library Maxwell Air Force Base, Alabama ATTN: CR-4582	Department of the Army Office of Chief Signal Officer Washington 25, D.C. ATTN: SIGGD	U.S. Naval Ordnance Laboratory White Oak, Silver Spring 19, Maryland ATTN: The Library
Commander Air Force Missile Test Center Patrick Air Force Base, Florida ATTN: Technical Library	Department of the Army Office of Chief Signal Officer Engineering and Technical Division Washington 25, D.C. ATTN: SIGNET-5	Commander U.S. Naval Ordnance Test Station (Code 753) China Lake, California ATTN: Tech. Library Branch
Headquarters, USAF Director of Research and Development Washington 25, D.C. ATTN: AFDRD	Diamond Ordnance Fuze Laboratories Washington 25, D.C. ATTN: Guided Missile Fuze Library R.D. Rethcher, Chief Microwave Dev. Section	Librarian U.S. Naval Postgraduate School Monterey, California
Commander Wright Air Development Center Arlington-Patterson Air Force Base, Ohio ATTN: Mr. Walter J. Fortune, WCLRS-6	(10) Armed Services Tech. Information Agency Arlington Hall Station Arlington 12, Virginia ATTN: DSC-SA	Director U.S. Naval Research Laboratory Washington 25, D.C. ATTN: Code 2021
Commander Wright Air Development Center Arlington-Patterson Air Force Base, Ohio ATTN: WCLNT-5, N. Draganjac	National Bureau of Standards Boulder Laboratories Library Boulder, Colorado ATTN: Mrs. Victoria S. Barker, Lib.	Director U.S. Naval Research Laboratory Washington 25, D.C. ATTN: J. Bohnert, Code 5250
Commander Wright Air Development Center Arlington-Patterson Air Force Base, Ohio ATTN: WCLRE-5, Wave Propagation Section	National Bureau of Standards Department of Commerce Washington 25, D.C. ATTN: Dr. A.G. McNish	Commander U.S. Navy Underwater Sound Laboratory Fort Trumbull, New London Conn. ATTN: Mr. C.M. Dunn
Air Technical Intelligence Center Wright-Patterson Air Force Base, Ohio ATTN: AFQIN-4B1a	National Bureau of Standards Department of Commerce Washington 25, D.C. ATTN: Gustave Shapiro Electronic Miniaturization Lab. Engineering Electronics Section Electricity and Electronics Div.	Chief of Naval Research Navy Department, Washington 25, D.C. ATTN: Code 427
(2) Commander Rome Air Development Center Griffiss Air Force Base, New York ATTN: RCSST-3	Director National Bureau of Standards Central Radio Propagation Lab. Boulder, Colorado ATTN: Division 14.0	(2) Commanding Officer and Director U.S. Navy Electronics Lab. Library San Diego 52, California
Commander Rome Air Development Center Griffiss Air Force Base, New York ATTN: D. Mather, RCERT	(2) Office of Technical Services Department of Commerce Washington 25, D.C. ATTN: Technical Reports Section	Chief, Bureau of Ordnance Navy Department, Washington 25, D.C. ATTN: Technical Library, Code Ad3 Mr. H. B. Rex, Code ReSh
Rome Air Development Center Griffiss Air Force Base, New York ATTN: Dr. John Burgess, RCER	Library, Div. of Public Documents U.S. Government Printing Office Washington 25, D.C.	Chief, Bureau of Ordnance Navy Department, Washington 25, D.C. ATTN: Mr. C.H. Jackson, Code Re9a
Commander Holloman Air Research and Development Center Holloman Air Force Base, New Mexico ATTN: HDOI	Director, National Security Agency Washington 25, D.C. ATTN: R/D (331)	Chief, Bureau of Ordnance Navy Department, Washington 25, D.C. ATTN: L.E. Wallingford, Code RelF
Director Ballistics Research Laboratory Aberdeen Proving Ground, Maryland ATTN: Ballistics Measurement Laboratory	(2) Hq. Air Force Cambridge Research Center Air Research and Development Command Laurence G. Hanscom Field Bedford, Mass. ATTN: CROOFR-2 - P. Condon	Chief, Bureau of Aeronautics Department of the Navy, Washington 25, D.C. ATTN: Code Aer-EL-931
Ballistic Research Laboratories Aberdeen Proving Ground, Maryland ATTN: Technical Information Branch Victor W. Richard	(5) Hq. Air Force Cambridge Research Center Air Research and Development Command Laurence G. Hanscom Field Bedford, Mass. ATTN: CROOFR - J. Armstrong	(2) Commander U.S. Naval Air Development Center Johnsville, Pennsylvania
Director Evans Signal Laboratory Belmar, New Jersey ATTN: Mr. O.C. Woodyard	(3) Hq. Air Force Cambridge Research Center Air Research and Development Command Laurence G. Hanscom Field Bedford, Mass. ATTN: CRRD - R.E. Riatt	Commander U.S. Naval Air Test Center Patuxent River, Maryland ATTN: ET-315, Antenna Section
Director Communications and Electronics Hq., U.S. Air Force Washington 25, D.C. ATTN: AFDAC s/e		Naval Ordnance Laboratory Corona, California ATTN: Library
		U.S. Naval Ordnance Laboratory Dept. of the Navy Corona, California ATTN: Dr. Robert L. Conger

Airborne Instruments Laboratory, Inc.  
160 Old Country Road  
Mineola, New York  
Attn: Dr. E.G. Fubini, Director  
Research and Engineering Division

Aircom, Inc.  
354 Main Street  
Winthrop, Mass.

American Machine and Foundry Company  
Electronics Division  
1085 Commonwealth Avenue  
Boston 15, Mass.  
Attn: Mrs. Rita Moravcsik, Librarian

Andrew Alford Consulting Engineers  
299 Atlantic Avenue  
Boston 10, Mass.

Avion Division  
ACF Industries, Inc.  
800 North Pitt Street  
Alexandria, Virginia  
Attn: Technical Library

Battelle Memorial Institute  
505 King Avenue  
Columbus 1, Ohio  
Attn: Dr. Bernard H. List

Bell Aircraft Corporation  
P.O. Box One, Buffalo 5, N.Y.  
Attn: Mrs. J. Mulcahey, Tech. Lib.

Bell Telephone Laboratories, Inc.  
Whippany Laboratory, Whippany, N.J.  
Attn: Technical Information Lib.

Bendix Aviation Corporation  
Pacific Division, 11600 Sherman Way  
North Hollywood, Calif.  
Attn: J.R. Breninger, Engr. Librarian

Bendix Radio, Div. of Bendix Aviation Corp.  
East Joppa Road, Towson 4, Maryland  
Attn: Dr. D.M. Allison, Jr.  
Dir. Engineering and Research

Bjorksten Research Laboratories, Inc.  
P.O. Box 265  
Madison 1, Wisconsin  
Attn: F.B. Korsgard, Treasurer

Boeing Airplane Company  
Pilotless Aircraft Division  
Seattle 24, Washington  
Attn: R.R. Barber, Library Supervisor

Boeing Airplane Company  
Wichita Division Engineering Library  
Wichita 1, Kansas  
Attn: Kenneth C. Knight, Librarian

Boeing Airplane Company  
Seattle Division  
Seattle 14, Washington  
Attn: E.T. Allen, Library Supervisor

Convair, A Division of General Dynamics Corp.  
Fort Worth, Texas  
Attn: K.O. Brown, Division Research Lib.

Convair, A Div. of General Dynamics Corp.  
San Diego 12, California  
Attn: Mrs. Dora B. Burke, Engr. Lib.

Cornell Aero. Laboratory, Inc.  
4455 Genesee Street  
Buffalo 21, New York  
Attn: Elma T. Evans, Librarian

Dalmo Victor Company  
A Division of Textron, Inc.  
1515 Industrial Way  
Belmont, California  
Attn: Mary Ellen Addams, Tech. Lib.

Dorne and Margolin, Inc.  
30 Sylvester Street  
Westbury, Long Island, N.Y.

Douglas Aircraft Company, Inc.  
P.O. Box 200  
Long Beach 1, California  
Attn: Engineering Lib. (C-250)

Douglas Aircraft Company, Inc.  
827 Lapham Street  
El Segundo, California  
Attn: Engineering Library

Douglas Aircraft Company, Inc.  
3000 Ocean Park Boulevard  
Santa Monica, California  
Attn: Eq. Sec. Ref. Files, Eq. Eng. A250

Douglas Aircraft Corporation  
2000 North Memorial Drive  
Tulsa, Oklahoma  
Attn: Engineering Library, D-250

Electronic Communication, Inc.  
St. Paul Street at University Parkway  
The Maryland Apartments  
Baltimore 18, Maryland  
Attn: Dr. D.D. King, Vice-President

Electronic Defense Laboratory  
P.O. Box 205  
Mountain View, California  
Attn: Library

Electronics Research, Inc.  
2300 N. New York Avenue  
Evansville, Indiana  
Attn: B.H. Baldrige, Engr. Supervisor

Emerson and Cuming, Inc.  
869 Washington Street  
Canton, Mass.  
Attn: W.R. Cuming

The Emerson Electric Mfg. Company  
8100 Florissant Avenue  
St. Louis 21, Missouri  
Attn: Mr. E. Breslin, Librarian

Fairchild Aircraft  
Division Fairchild Eng. and Airplane Corp.  
Hagerstown, Maryland  
Attn: Library

Farnsworth Electronics Company  
Fort Wayne 1, Indiana  
Attn: Technical Library

Federal Telecommunication Laboratories  
500 Washington Avenue  
Nutley 10, New Jersey  
Attn: Technical Library

Gabriel Electronics  
Division of Gabriel Company  
135 Crescent Road  
Needham Heights 94, Mass.  
Attn: Mr. Steven Galagan

General Electric Company  
Ithaca, New York  
Attn: J.B. Travis,  
Advanced Electronics Center

General Electric Company  
Electronics Park  
Syracuse, New York  
Attn: Bessie Fletcher, Documents Library  
Bldg. 3-143A

General Precision Laboratory, Inc.  
63 Bedford Road  
Pleasantville, New York  
Attn: Mrs. Mary G. Herbst, Librarian

Goodyear Aircraft Corp.  
1210 Massillon Road  
Akron 15, Ohio  
Attn: Library D/120 Plant A

Granger Associates  
Electronic Systems  
966 Commercial Street  
Palo Alto, California  
Attn: John V.N. Granger, Pres.

Grumman Aircraft Engineering Corp.  
Bethpage, Long Island, N.Y.  
Attn: Mrs. A.M. Gray, Librarian  
Engineering Library, Plant No. 5

The Hallcrafters Co.  
4401 West 5th Avenue  
Chicago 24, Illinois  
ATTN: L. La Gioia, Librarian

Hoffman Laboratories, Inc.  
3740 South Grand Avenue  
Los Angeles 7, California  
Attn: Engineering Library

Hughes Aircraft Co., Antenna Dept.  
Microwave Laboratory, Bldg. 12, Rm. 2617  
Culver City, California  
Attn: M.D. Adcock

Hughes Aircraft Co.  
Florence and Teale Streets  
Culver City, California  
Attn: Dr. L.C. Van Atta, Assoc. Dir.  
Research Laboratories

Hycon Eastern, Inc.  
75 Cambridge Parkway  
Cambridge, Mass.  
Attn: Mrs. Lois Saulowitz, Tech. Lib.

International Business Machines Corp.  
Military Products Division  
590 Madison Avenue  
New York 22, New York  
Attn: Mr. C.F. McElwain, General Manager

International Business Machines Co.  
Military Products Div.  
Oswego, New York  
Attn: Mr. D.I. Marr, Librarian  
Dept. 459

International Resistance Company  
401 N. Broad Street  
Philadelphia 8, Pa.  
Attn: Research Library

Jansky and Bailey, Inc.  
1339 Wisconsin Avenue, N.W.  
Washington 7, D.C.  
Attn: Mr. Delmer C. Ports

Dr. Henry Jasik, Consulting Engr.  
298 Shames Drive,  
Brush Hollow Industrial Park  
Westbury, New York

Martin Katzin and Company, Consultants  
711 14th Street, N.W.  
Washington 5, D.C.

Lockheed Aircraft Corp.  
Missile Systems Division  
7701 Woodley Avenue  
Van Nuys, California  
Attn: Van Nuys Lib. Bldg. 902

Lockheed Aircraft Corp.  
California Division Engr. Lib.  
Dept. 72-25, Plant A-1, Bldg. 63-1  
Burbank, California  
Attn: N.C. Harnois

The Glenn L. Martin Company  
Denver Division  
Waterton, Colorado  
Attn: William V. Foley,  
Test and Reliability  
Mail No. 2-38

The Glenn L. Martin Company  
Baltimore 3, Maryland  
Attn: Engr. Lib., Antenna Design Group

Maryland Electronic Mfg. Corp.  
5009 Calvert Road  
College Park, Maryland  
Attn: Mr. H. Warren Cooper

Mathematical Reviews  
190 Hope Street  
Providence 6, Rhode Island

The W.L. Maxson Corporation  
460 West 34th Street  
New York, N.Y.  
Attn: Miss Dorothy Clark

McDonnell Aircraft Corp.  
Lambert Saint-Louis Municipal Airport  
Box 516, St. Louis 3, Missouri  
Attn: R.D. Dietrich, Engr. Library

McMillan Laboratory, Inc.  
Brownville Avenue  
Ipswich, Massachusetts  
Attn: Security Officer, Document Rm.

Melpar, Inc.  
3000 Arlington Blvd.  
Falls Church, Va.  
Attn: Engineering Tech. Lib.

Microwave Development Laboratory  
90 Broad Street  
Babson Park 57, Mass.  
Attn: N. Tucker, General Manager

Microwave Radiation Company  
19223 South Hamilton St.  
Gardena, California  
Attn: Mr. Morris J. Ehrlich, Pres.

Naval Industrial Reserve Aircraft Plant  
P.O. Box 5907  
Dallas, Texas  
Attn: Mr. H.S. White, Librarian

Northrop Aircraft, Inc.  
Hawthorne, California  
Attn: Mr. E.A. Freitas, Lib. Dept. 3145

North American Aviation, Inc.  
12214 Lakewood Blvd.  
Downey, California  
Attn: Engr. Lib. 495-115

North American Aviation, Inc.  
Los Angeles International Airport  
Los Angeles 45, California  
Attn: Engineering Technical File

Page Communications Engineers, Inc.  
710 Fourteenth Street, Northwest  
Washington 5, D.C.  
Attn: Librarian

Philco Corporation Res. Division  
Branch Library  
4700 Wissachickon Avenue  
Philadelphia 44, Pa.  
Attn: Mrs. Dorothy S. Collins

Pickard and Burns, Inc.  
240 Highland Avenue  
Needham 94, Mass.  
Attn: Dr. J.T. deBettencourt

Polytechnic Research and Dev. Co., Inc.  
202 Tillary Street  
Brooklyn 1, New York  
Attn: Technical Library

Radiation Engineering Laboratory  
Main Street  
Maynard, Mass.  
Attn: Dr. John Ruze

Radiation, Inc.  
P.O. Drawer 37  
Melbourne, Florida  
Attn: Tech. Library, Mr. M.L. Cox,  
Assistant Project Engineer

Radio Corp. of America  
RCA Laboratories  
Rocky Point, New York  
Attn: P.S. Carter, Lab. Library

RCA Laboratories  
David Sarnoff Research Center  
Princeton, New Jersey  
Attn: Miss Fern Cloak, Lib., Res. Library

RCA Defense Electronic Products  
Camden 2, New Jersey  
Attn: Clarence A. Gunther, Chief Def. Engr.  
DEP - Bldg. 15, Floor 7

(2)The Ramo-Wooldridge Corporation  
5730 Arbor Vitae Street  
Los Angeles 45, California  
Attn: Margaret C. Whitnah, Chief Lib.

Hoover Microwaves  
9592 Baltimore Avenue  
College Park, Maryland

The Rand Corporation  
1700 Main Street  
Santa Monica, California  
Attn: Library - via: SBAMA Liaison Office

Rantec Corporation  
Calabasas, California  
Attn: Grace Kessler, Office Manager

Raytheon Manufacturing Company  
Missile Systems Division  
Bedford, Mass.  
Attn: Mr. Irving Goldstein

Raytheon Manufacturing Company  
Wayland Laboratory  
Wayland, Mass.  
Attn: J.E. Walsh, Dept. 8974,  
Antenna Design Radar

Raytheon Manufacturing Company  
Wayland Laboratory  
Wayland, Mass.  
Attn: Miss Alice G. Anderson, Librarian

Republic Aviation Corporation  
Farmingdale, Long Island, N.Y.  
Attn: R.E. Fidoten, Engineering Lib.

Rheem Manufacturing Company  
Research and Development  
9236 East Hall Road  
Downey, California  
Attn: J.C. Joeger

Ryan Aeronautical Company  
Lindbergh Field  
San Diego 12, California  
Attn: Library

Sage Laboratories, Inc.  
30 Guinan Street  
Waltham 54, Mass.

Sanders Associates, Inc.  
95 Canal Street  
Nashua, New Hampshire  
Attn: N.R. Wild, Library

Sandia Corporation, Sandia Base  
P.O. Box 5800, Albuquerque, New Mexico  
Attn: Classified Document Division

Sperry Gyroscope Company  
Great Neck, L.I., New York  
Attn: Florence W. Turnbull, Engr. Lib.

Stanford Research Institute  
Document Center  
Menlo Park, California  
Attn: Mary Lou Fields--Requisitions

Sylvania Electric Products, Inc.  
100 First Avenue  
Waltham 54, Mass.  
Attn: Charles A. Thornhill, Reports Lib.  
Waltham Laboratories Library

Systems Laboratories Corporation  
15016 Ventura Boulevard  
Sherman Oaks, California  
Attn: Dr. W.C. Hoffman

Technical Research Group  
17 Union Square West  
New York 3, N.Y.  
Attn: M.L. Henderson, Librarian

A.S. Thomas, Inc.  
161 Devonshire Street  
Boston 10, Mass.  
Attn: A.S. Thomas, President

Westinghouse Electric Corp.  
2519 Wilkens Avenue  
Baltimore 3, Maryland  
Attn: Kent M. Mack, Engr. Lib.

Wheeler Laboratories, Inc.  
122 Cutter Mill Road  
Great Neck, New York  
Attn: Mr. Harold A. Wheeler

Zenith Plastics Company  
Box 91, Gardena, California  
Attn: Mr. S.S. Oleesky

Library, Geophysical Institute of the  
University of Alaska  
College, Alaska

University of California  
Berkeley 4, California  
Attn: S. Silver, Prof. Engr. Science  
Division of Elec. Engr.  
Electronics Research Lab.

Electronics Research Laboratory  
University of California  
332 Cory Hall, Berkeley 4, Calif.  
Attn: J.R. Whinnery, Prof.

California Institute of Technology  
Jet Propulsion Laboratory  
4800 Oak Grove Drive  
Pasadena, California  
Attn: I.E. Newlan

California Institute of Technology  
1201 E. California Street  
Pasadena, California  
Attn: Dr. Charles H. Papas

Carnegie Institute of Technology  
Schenley Park  
Pittsburgh 13, Pennsylvania  
Attn: Albert E. Heina

Cornell University  
School of Electrical Engineering  
Ithaca, New York  
Attn: Prof. G.C. Dalman

University of Florida  
College of Engineering  
Gainesville, Florida  
Attn: Prof. M.H. Latour,  
Engineering Sciences Lib.

Georgia Institute of Technology  
Engineering Experiment Station  
Atlanta, Georgia  
Attn: Mrs. J.H. Crosland  
Georgia Tech. Library

Harvard University  
Technical Reports Collection  
Gordon McKay Library, 303A Pierce Hall  
Oxford Street, Cambridge 38, Mass.  
Attn: Mrs. E.L. Hufschmidt, Lib.

Harvard College Observatory  
60 Garden Street  
Cambridge, Mass.  
Attn: Dr. Fred L. Whipple

University of Illinois  
Serials Dept. - 2205 Library  
Urbana, Illinois

Electrical Engineering Res. Lab.  
University of Illinois  
Urbana, Illinois  
Attn: Antenna Lab.

The Johns Hopkins University  
Homewood Campus  
Department of Physics  
Baltimore 18, Maryland  
Attn: Professor Donald E. Kerr

The Johns Hopkins University  
8621 Georgia Avenue  
Silver Spring, Maryland  
Attn: Mr. George L. Seielstad,  
Applied Physics Lab.

Massachusetts Institute of Technology  
Research Laboratory of Electronics  
Document Room 20B-221  
Cambridge 39, Mass.

Massachusetts Institute of Tech.  
Lincoln Laboratory  
P.O. Box 73  
Lexington 73, Mass.  
Attn: Henry Straus, Document Rm. A-229

University of Michigan  
Electronic Defense Group  
Engineering Research Institute  
Ann Arbor, Michigan  
Attn: J.A. Boyd, Supervisor

The University of Michigan  
Willow Run Laboratories  
Willow Run Airport  
Ypsilanti, Michigan  
Attn: K.M. Siegel, Head  
Theory and Analysis Dept.

University of Michigan  
Willow Run Laboratories  
Engineering Research Institute  
Willow Run Airport  
Ypsilanti, Michigan  
Attn: Librarian

The University of Minnesota  
Minneapolis 14, Minnesota  
Attn: Mr. Robert H. Stumm, Library

Northwestern University  
Microwave Laboratories  
Evanston, Illinois  
Attn: Professor R.E. Peam

Ohio State University  
Research Foundation  
Columbus 10, Ohio  
Attn: Dr. T.E. Tice  
Dept. of Electrical Engr.

The University of Oklahoma  
Norman, Oklahoma  
Attn: Prof. C.L. Farrar  
Chairman Electrical Engr.

Polytechnic Institute of Brooklyn  
Microwave Research Institute  
55 Johnson Street  
Brooklyn 1, New York  
Attn: Mr. A.E. Laemmel

Syracuse University  
Research Institute  
Collendale D-6  
Syracuse 10, New York  
Attn: Dr. C.S. Grove, Jr.  
Director of Engr. Research

The University of Texas  
Electrical Engr. Research Lab.  
P.O. Box 8026, University Station  
Austin 12, Texas  
Attn: J.R. Gerhardt, Assistant Dir.

The University of Texas  
Defense Research Lab.  
Austin 12, Texas  
Attn: Claude W. Horton, Phys. Lib.

The University of Toronto  
Department of Electrical Engineering  
Toronto, Canada  
Attn: Prof. G. Sinclair

LTI Research Foundation  
P.O. Box 709  
Lowell, Mass.  
Attn: Dr. Charles R. Mingins

University of Washington  
Dept. of Electrical Engineering  
Seattle 5, Washington  
Attn: G. Held, Associate Professor

Polytechnic Institute of Brooklyn  
Microwave Research Institute  
55 Johnson Street  
Brooklyn 1, New York  
Attn: Dr. A.A. Oliner

Electronica Division  
Rand Corporation  
1700 Main Street  
Santa Monica, California  
Attn: Dr. Robert Kalaba

National Bureau of Standards  
Washington, D.C.  
Attn: Dr. W.K. Saunders

Applied Mathematics and Statistics Lab.  
Stanford University  
Stanford, California  
Attn: Dr. Albert H. Bowker

Dept. of Physics and Astronomy  
Michigan State University  
East Lansing, Michigan  
Attn: Dr. A. Leitner

Ionosphere Research Laboratory  
Pennsylvania State College  
State College, Pennsylvania  
Attn: Prof. A.H. Waynick, Director

Institute of Mathematical Sciences  
25 Waverly Place  
New York 3, New York  
Attn: Mrs. Joan Segal, Librarian

School of Electrical Engineering  
Purdue University  
Lafayette, Indiana  
Attn: Professor F.V. Schultz

University of Tennessee  
Knoxville, Tennessee  
Attn: Dr. Fred A. Picken

California Institute of Tech.  
1201 E. California Street  
Pasadena, California  
Attn: Dr. A. Erdelyi

Wayne University  
Detroit, Michigan  
Attn: Professor A.F. Stevenson

Mathematics Department  
Stanford University  
Stanford, California  
Attn: Dr. Harold Levine

University of Minnesota  
Minneapolis 14, Minnesota  
Attn: Prof. Paul C. Rosenbloom

Department of Mathematics  
Stanford University  
Stanford, California  
Attn: Professor Bernard Epstein

Applied Physics Laboratory  
The Johns Hopkins University  
8621 Georgia Avenue  
Silver Spring, Maryland  
Attn: Dr. B.S. Gourary

(2) Exchange and Gift Division  
The Library of Congress  
Washington 25, D.C.

Electrical Engineering Dept.  
Massachusetts Institute of Technology  
Cambridge 39, Massachusetts  
Attn: Dr. L.J. Shu

Nuclear Development Associates, Inc.  
5 New Street  
White Plains, New York  
Attn: Library

Lebanon Valley College  
Annville, Pennsylvania  
Attn: Prof. B.H. Bissinger

Dept. of Physics  
University of Pittsburgh  
Thaw Hall  
Pittsburgh 13, Pa.  
Attn: Dr. Edward Gerjuoy

Dept. of Physics  
Amherst College  
Amherst, Massachusetts  
Attn: Dr. Arnold Arons

California Institute of Tech.  
Electrical Engineering  
Pasadena, California  
Attn: Dr. Zohrab A. Kaprielian

Dr. Rodman Doll  
311 W. Cross St.  
Ypsilanti, Michigan

California Institute of Technology  
Pasadena 4, California  
Attn: Mr. Calvin Wilcox

Mr. Robert Brockhurst  
Woods Hole Oceanographic Institute  
Woods Hole, Massachusetts

National Bureau of Standards  
Boulder, Colorado  
Attn: Dr. R. Callet

Mrs. Jane Scanlon  
224-03 67th Avenue  
Bayside 64, New York

Dr. Solomon L. Schwebel  
3689 Louis Road  
Palo Alto, California

University of Minnesota  
The University Library  
Minneapolis 14, Minnesota  
Attn: Exchange Division

Professor Bernard Friedman  
Dept. of Mathematics  
University of California  
Berkeley, California

Lincoln Laboratory  
Massachusetts Institute of Tech.  
P.O. Box 73  
Lexington 73, Massachusetts  
Attn: Dr. Shou Chin Wang, Rm. C-351

Melpar, Inc.  
3000 Arlington Boulevard  
Falls Church, Virginia  
Attn: Mr. K.S. Kelleher, Section Head

Hq. Air Force Cambridge Res. Center  
Laurence G. Hanscom Field  
Bedford, Massachusetts  
Attn: Lt. Mark J. Beran

Electronics Research Directorate  
Hq. Air Force Cambridge Res. Center  
Laurence G. Hanscom Field  
Bedford, Massachusetts  
Attn: Dr. Philip Newman, CRRK

Hq. Air Force Cambridge Res. Center  
Laurence G. Hanscom Field  
Bedford, Massachusetts  
Attn: Mr. Francis J. Zucker

Mr. N.C. Gerson  
Trapelo Road  
S. Lincoln, Massachusetts

Systems Development Corp.  
2400 Colorado  
Santa Monica, California  
Attn: Dr. Richard B. Barrar

Columbia University  
Hudson Laboratories  
P.O. Box 239  
145 Palisade Street  
Dobbs Ferry, New York  
Attn: Dr. M.W. Johnson

Institute of Fluid Dynamics  
and Applied Math, Univ. of Maryland  
College Park, Maryland  
Attn: Dr. Elliott Montroll

Dept. of Electrical Engineering  
Washington University  
Saint Louis 5, Missouri  
Attn: Prof. J. Van Bladel

Dept. of the Navy  
Office of Naval Res. Branch Office  
1030 E. Green Street  
Pasadena 1, California

Brandeis University  
Waltham, Massachusetts  
Attn: Library

General Electric Company  
Microwave Laboratory  
Electronics Division  
Stanford Industrial Park  
Palo Alto, California  
Attn: Library

Smyth Research Associates  
3930 4th Avenue  
San Diego 3, California  
Attn: Dr. John B. Smyth

Electrical Engineering  
California Inst. of Technology  
Pasadena, California  
Attn: Dr. Georges Weill

Naval Research Laboratory  
Washington 25, D.C.  
Attn: Dr. Henry J. Passerini  
Code 5278 A

Dr. George Kear  
5 Culver Court  
Orinda, Calif.

Brooklyn Polytechnic  
85 Livingston Street  
Brooklyn, New York  
Attn: Prof. Nathan Marcuvitz

Dept. of Electrical Engineering  
Brooklyn Polytechnic  
85 Livingston Street  
Brooklyn, New York  
Attn: Dr. Jerry Shmoys

Dept. of Mathematics  
University of New Mexico  
Albuquerque, New Mexico  
Attn: Dr. I. Kolodner

Mathematics Dept.  
Polytechnic Institute of Brooklyn  
Johnson and Jay Streets  
Brooklyn, New York  
Attn: Dr. Harry Hochstadt











